Math 401 Midterm Review

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March 2025

Exercise 1: Is the kernel of a homomorphism of groups always a normal subgroup? Is the image of a homomorphism of groups always a normal subgroup? Give a proof or counterexample for each statement.

Exercise 2:

1. Prove that any group G acts on itself by conjugation. That is, the map $G \times G \to G$, $(g, h) \mapsto ghg^{-1}$ defines a group action.

2. We define the conjugacy class $[h]$ of an element $h \in G$ to be the set $\{ghg^{-1} \mid g \in G\}$ consisting of all elements in G conjugate to h. Is $[h]$ always a subgroup of G?

3. Does right multiplication $(g, h) \mapsto hg$ define an action of G on itself? How about right multiplication by the inverse: $(g, h) \mapsto hg^{-1}$?

Exercise 3: Let G be a group and let $Z(G)$ be the center of G. Prove that if $G/Z(G)$ is cyclic then G is abelian.

Exercise 4:

1. Describe all homomorphisms $\mathbb{Z} \to A_5$.

2. Describe all homomorphisms $\mathbb{Z}/5\mathbb{Z} \to \mathbb{Z}/15\mathbb{Z}$.

3. Describe all homomorphisms $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}$.

Exercise 5: Let G be a group with *normal* subgroups H and K. Further, suppose that $H \cap K = \{e\}$ and $G = HK$. Prove that $G \cong H \times K$.

Exercise 6: Prove that $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z}$ but $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ is not isomorphic to $\mathbb{Z}/9\mathbb{Z}$.

Exercise 7:

1. Let G be a finite group and $n \in \mathbb{N}$. Suppose that $|G|$ is coprime to n. Prove that the map $\phi : G \to G$, $g \mapsto g^n$ is surjective. Hint: Use Bezout's lemma to write $nx + |G|y = 1$ for some $x, y \in \mathbb{Z}$.

2. In fact, the converse statement is true. That is, if ϕ as given above is surjective, then $|G|$ is coprime to n. One can proceed by contradiction to get a proof. Suppose that $d = \gcd(n, |G|) > 1$. Then, there exists a prime p dividing d. By Cauchy's theorem, there exists $x \in G$ of order p. We can write $x = y^n$ for some $y \in G$. Since ϕ is a surjection between finite sets of equal cardinality, it is also an injection. Thus, $1 = x^p = (y^p)^n \implies y^p = 1$. Can you finish the proof by contradiction?

3. Is ϕ guaranteed to be a homomorphism?

Exercise 8: Give a group structure on the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and prove that the quotient group $\mathbb{R}/\mathbb{Z}$ is isomorphic to S^1 with this group structure. Hint: Consider parameterizing S^1 by the function $e^{2\pi it}$.

Exercise 9: Prove that $(\mathbb{Q}, +)$ is not cyclic. In fact, $\mathbb{Q}$ is not finitely generated. Hint: Consider the uniqueness of prime factorizations.

Exercise 10: Let G be a group and $g \in G$. Prove that the map $x \mapsto gx$ defines a bijection on G. Is this map a homomorphism?

Exercise 11: If G and H are groups, then we can form their direct product $G \times H$. If $G_1 \leq G$ and $H_1 \leq H$, then $G_1 \times H_1 \leq G \times H$. However, not all subgroups arise in this way. Give an example of a subgroup of a product of groups that is not a product of subgroups.

Exercise 12: Prove that a group of prime order is cyclic.

Exercise 13: Prove that $|A_n| = \frac{n!}{2}$ and use this fact to show that A_n is a normal subgroup of S_n.

Exercise 14: For any group G, define the commutator subgroup $[G, G]$ to be the subgroup generated by elements of the form $g^{-1}h^{-1}gh$ where $g, h \in G$. Give an example of a group whose commutator subgroup is not $\{e\}$.

Exercise 15: Suppose that G is a group and $\emptyset \neq H \subseteq G$ is finite. Prove that H is a subgroup of G if and only if the product of two elements of H remains in H. Is this true for infinite subsets?